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EXISTENCE OF $\mathcal{N}$-INJECTORS IN A NOT CENTRAL NORMAL FITTING CLASS

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ABSTRACT

If we denote by $L(G)$ the semisimple radical of a group G, we prove in this paper that $\mathcal{L} = \{G \mid G = C_G(L(G))L(G)\}$ is a not central normal Fitting class. Moreover, all $\mathcal{L}$-groups have $\mathcal{N}$-injectors.

Introduction. Notation

All groups considered throughout this paper will be finite. $\mathcal{N}$ is the class of nilpotent groups. The concept of semisimple groups is taken from Gorenstein–Walter's paper ([2]). A Fitting class $\mathcal{F}$ is said to be central if $G/G_{\mathcal{F}}$ is abelian for every group G ([6]). The remainder notation is standard and is taken mainly from ([3]).

A group G is $\mathcal{N}$-constrained if $C_G(F(G)) \leqq F(G)$ ([7]). The class of $\mathcal{N}$-constrained groups is an extensible Fitting class that contains the solvable groups and a group G is $\mathcal{N}$-constrained if and only if $L(G) = 1$ ([5]), [8]).

A. Mann proves that a $\mathcal{N}$-constrained group has an unique conjugacy class of $\mathcal{N}$-injectors which are the maximal nilpotent subgroups containing $F(G)$. Blessenohl and Laue prove that all groups have an unique conjugacy class of $\mathcal{K}$-injectors, where $\mathcal{K}$ is the class of quasinilpotent groups, i.e. $\mathcal{K} = \{G \mid G = F(G)L(G) = F^*(G)\}$, moreover these injectors are the maximal-$\mathcal{K}$-subgroups containing $F^*(G)$. It is well known that $C_G(F^*(G)) \leqq F(G)$ for every group G ([4]).

The aim of this paper is mainly to prove the following results:

THEOREM A. *For every group G we have:*
(a) *If V is a $\mathcal{K}$-injector of G then $V \leqq G_{\mathcal{L}}$.*
(b) *If $G_{\mathcal{L}} \leqq U \leqq G$, $U \in \mathcal{L}$ then $U = L(U)G_{\mathcal{L}}$.*

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THEOREM B. *Let G be a $\mathscr{L}$-group, then G contains $\mathscr{N}$-injectors which are the product of an $\mathscr{N}$-injector of $L(G)$ and an $\mathscr{N}$-injector of $C_G(L(G))$.*

Notice that all $\mathscr{N}$-constrained groups are $\mathscr{L}$-groups.

Before proving the theorems we give the following results:

LEMMA 1. *Let N be a normal subgroup of the semisimple group G. Then either $N \leqq Z(G)$ or N' is semisimple nontrivial and $N'Z(N) = N$.*

PROOF. Since G is semisimple, $G/Z(G)$ is a direct product of nonabelian simple groups. As $NZ(G)/Z(G) \trianglelefteq G/Z(G)$, then either $N \leqq Z(G)$ or $NZ(G)/Z(G)$ is a direct product of nonabelian simple groups too. We assume the last case:

Hence $N/Z(N)$ is also a direct product of nonabelian simple groups and by Gorenstein–Walter's property, N' is semisimple and covers $N/Z(N)$.

LEMMA 2. *A group G contains $\mathscr{N}$-injectors if and only if $G/Z(G)$ contains $\mathscr{N}$-injectors.*

PROOF. Let H be an $\mathscr{N}$-injector of G and $G^*/Z(G) \trianglelefteq\trianglelefteq G/Z(G)$ then $H/Z(G) \cap G^*/Z(G) = (H \cap G^*)/Z(G)$. Since $H \cap G^*$ is an $\mathscr{N}$-maximal subgroup of G^* it follows that $(H \cap G^*)/Z(G)$ is an $\mathscr{N}$-maximal subgroup of $G^*/Z(G)$.

Conversely, assume that $H/Z(G)$ is an $\mathscr{N}$-injector of $G/Z(G)$ and $G^* \trianglelefteq\trianglelefteq G$. Let $H \cap G^* \leqq F \leqq G^*$, F nilpotent, then we have:

$$(H \cap G^*Z(G))/Z(G) \leqq FZ(G)/Z(G) \leqq G^*Z(G)/Z(G) \trianglelefteq\trianglelefteq G/Z(G),$$

hence $FZ(G) = (H \cap G^*)Z(G)$ and so

$$F = F \cap (H \cap G^*)Z(G) = (H \cap G^*)(F \cap Z(G)) \leqq H \cap G^*.$$

COROLLARY 1. *If G is a semisimple group then G contains $\mathscr{N}$-injectors.*

PROOF. Evidently the $\mathscr{N}$-maximal subgroups of a simple group are its $\mathscr{N}$-injectors. If G is a direct product of nonabelian simple groups $G = N_1 \times \cdots \times N_r$ and H_i is an $\mathscr{N}$-injector of N_i then $H_1 \times \cdots \times H_r$ is an $\mathscr{N}$-injector of G.

Let G be a semisimple group, then $G/Z(G)$ is a direct product of nonabelian simple groups ([2]), hence $G/Z(G)$ has $\mathscr{N}$-injectors and by Lemma 2 it follows that G has $\mathscr{N}$-injectors.

LEMMA 3. *For every group G, $C_G(L(G))$ is the $\mathscr{N}$-constrained radical of G.*

PROOF. As $L(C_G(L(G))$ is a semisimple normal subgroup of G thus $L(C_G(L(G)) \leqq L(G)$, hence $L(C_G(L(G))) \leqq Z(L(G))$, and since all semisimple groups are perfect groups we obtain $L(C_G(L(G))) = 1$ and so $C_G(L(G))$ is a $\mathcal{N}$-constrained group.

Assume that N is a $\mathcal{N}$-constrained normal subgroup of G, then $L(N) = 1$ and by Lemma 1 we obtain $[N, L(G)] = 1$.

LEMMA 4. *For every group G,*

$$F^*(G/Z(G)) = F^*(G)/Z(G).$$

PROOF. From ([4], X 13. 1–2–3) $\mathcal{K}$ is a Fitting Formation and trivially is closed for central extensions. If we denote $F^*(G/Z(G)) = M/Z(G)$ then $M/Z(M) \in \mathcal{K}$ and thus $M \in \mathcal{K}$, hence $M \leqq F^*(G)$.

PROPOSITION. *If $\mathcal{L} = \{G \mid G = C_G(L(G))L(G)\}$ then:*

(a) *$\mathcal{L}$ is a Fitting class.*

(b) *$G/Z(G) \in \mathcal{L}$ if and only if $G \in \mathcal{L}$.*

PROOF. (a) $S_n\mathcal{L} = \mathcal{L}$. It is enough to prove that every maximal normal of a $\mathcal{L}$-group is a $\mathcal{L}$-group too. Let G be a $\mathcal{L}$-group and N a maximal normal subgroup of G. If $L(G) \leq N$ we have:

$$L(G) \leqq N \leqq C_G(L(G))L(G)$$

so

$$N = N \cap C_G(L(G))L(G) = L(G)C_N(L(G)) = L(N)C_N(L(N))$$

and thus $N \in \mathcal{L}$.

If $L(G) \not\leqq N$ is $G = L(G)N$ and therefore $C_G(L(G)) \leqq N$, because in the contrary case $G = C_G(L(G))N = L(G)N$ and then G/N would be an abelian and semisimple group, hence G/N would be trivial.

Thus

$$N = N \cap C_G(L(G))L(G) = C_G(L(G))(N \cap L(G)).$$

By Lemma 3, $C_G(L(G)) = N \cap C_G(L(G)) = C_N(L(N))$, and by Lemma 1,

$$N = C_N(L(N))(N \cap L(G)) = C_N(L(N))L(N) \in \mathcal{L}.$$

$N_0\mathcal{L} = \mathcal{L}$. Let N_1, N_2 be normal subgroups of G and assume $N_i \in \mathcal{L}$, $i = 1,2$. Then $N_i = C_{N_i}(L(N_i))L(N_i)$, $i = 1,2$. By Lemma 3 we know that

$$C_{N_1N_2}(L(N_1N_2)) \cap N_i = C_{N_i}(L(N_i)), \qquad i = 1,2.$$

Therefore as $L(N_i) \leqq L(N_1N_2)$, $i = 1, 2$ we have

$$N_1N_2 = C_{N_1}(L(N_1))L(N_1)C_{N_2}(L(N_2))L(N_2) = C_{N_1N_2}(L(N_1N_2))L(N_1N_2),$$

i.e. N_1N_2 is a $\mathscr{L}$-group.

(b) Let $G/Z(G)$ be a $\mathscr{L}$-group, then

$$G/Z(G) = C_{G/Z(G)}(L(G/Z(G)))L(G/Z(G))$$
$$= C_{G/Z(G)}(L(G/Z(G))F^*(G/Z(G)).$$

By Lemma 4, $F^*(G/Z(G)) = F^*(G)/Z(G)$. We denote $M/Z(G) = C_{G/Z(G)}(L(G/Z(G)))$, since the class of the $\mathscr{N}$-constrained groups is extensible and by Lemma 3 we have $M = C_G(L(G))$. Thus $G = C_G(L(G))F^*(G) = C_G(L(G))L(G)$.

Conversely if $G = C_G(L(G))L(G)$ then

$$G/Z(G) = C_G(L(G))/Z(G)F^*(G)/Z(G),$$

thus

$$G/Z(G) = C_{G/Z(G)}(L(G/Z(G)))F^*(G/Z(G)),$$

because quotients of $\mathscr{N}$-constrained groups by central subgroups are $\mathscr{N}$-constrained too, and by Lemma 4.

Proof of Theorem A.

(a) Let V be a $\mathscr{K}$-injector of G, then $V = F(V)L(V)$ and $F(G)L(G) \leqq V$ thus

$$L(G)F(V)/F(V) = F(G)L(G)F(V)/F(V) \trianglelefteq V/F(V).$$

But as $V/F(V)$ is a direct product of nonabelian simple groups, $F(V)L(G)/F(V)$ is a direct factor of $V/F(V)$. Assume

$$V/F(V) = L(G)F(V)/F(V) \times K/F(V)$$

whence $[K, L(G)] \leqq F(V)$ and consequently $[L(G), K, L(G)] = 1$, it follows that $L(G)$ centralizes $[L(G), K]$, whence $L(G)$ centralizes K by the three-subgroups lemma, thus $K \leqq C_G(L(G))$. Now, as $[F(V), L(V)] = 1$ and $L(G) \leqq L(V)$ is $F(V) \leqq C_G(L(G))$ whence

$$V = L(G)F(V)K \leqq L(G)C_G(L(G)) = G_{\mathscr{L}}.$$

(b) Assume $G_{\mathscr{L}} \leqq U \leqq G$ and $U \in \mathscr{L}$, then

$$C_G(L(G))L(G) \leqq C_U(L(U))L(U).$$

As $L(G) \leqq L(U)$ it follows that

$$C_U(L(U)) \leqq C_U(L(G)) \leqq C_G(L(G)) \leqq U$$

thus $C_U(L(U)) = C_G(L(G))$ by Lemma 3. We have:

$$U = L(U)C_U(L(U)) = L(U)C_G(L(G)) = L(U)L(G)C_G(L(G)) = L(U)G_{\mathscr{L}}.$$

Corollary 2. *For every group G, $G_{\mathscr{L}}$ is a $\mathscr{L}$-maximal subgroup of G. Therefore $\mathscr{L}$ is a normal Fitting class. Moreover $\mathscr{L}$ is not central.*

Proof. Assume $G_{\mathscr{L}} \leqq U \leqq G$ with $U \in \mathscr{L}$, then by Theorem (b) we know that $U = L(U)G_{\mathscr{L}}$. Now since $F^*(G) \leqq F(U)L(U)$, there exists a $\mathscr{K}$-injector V of G, containing $L(U)$, hence

$$U = L(U)G_{\mathscr{L}} \leqq VG_{\mathscr{L}} = G_{\mathscr{L}}$$

by Theorem (a).

$\mathscr{L}$ is not central because if we take $G = A_5 \wr A_5$ then $L(G) = A_5$, $C_G(L(G)) = 1$ and $G/G_{\mathscr{L}} \cong A_5$ is not abelian.

Proof of Theorem B.

By induction on the order of G.

If $Z(G) \neq 1$ then by Proposition and Lemma 3

$$\begin{aligned} G/Z(G) &= C_{G/Z(G)}(L(G/Z(G)))L(G/Z(G)) \\ &= C_G(L(G))/Z(G)L(G)Z(G)/Z(G). \end{aligned}$$

If K is an $\mathcal{N}$-injector of $L(G)$ and H is an $\mathcal{N}$-injector of $C_G(L(G))$, then by Lemma 2, $KZ(G)/Z(G)$ and $H/Z(G)$ are $\mathcal{N}$-injectors of $L(G)Z(G)/Z(G)$ and $C_G(L(G))/Z(G)$ respectively, hence by induction $HK/Z(G)$ is an $\mathcal{N}$-injector of $G/Z(G)$ and again, by Lemma 2, HK is an $\mathcal{N}$-injector of G. Therefore we can suppose that $Z(G) = 1$, thus $Z(L(G)) = 1$, consequently $G = L(G) \times C_G(L(G))$ and $G/L(G) \cong C_G(L(G))$. We note that $HL(G)/L(G)$ is an $\mathcal{N}$-injector of $G/L(G)$. First, we prove that $V = HK$ is an $\mathcal{N}$-maximal subgroup of G. In fact, assume $V \leqq V_1 \leqq G$, V_1 nilpotent, then $HL(G)/L(G) \leqq V_1L(G)/L(G)$ and so $HL(G) = V_1L(G)$ thus

$$H \leqq V \leqq V_1 \leqq HL(G),$$

and since K is an $\mathcal{N}$-injector of $L(G)$, we have

$$V_1 = H(V_1 \cap L(G)) = HK = V.$$

Now let G^* be a subnormal subgroup of G. By Proposition, part (a) is $G^* \in \mathcal{L}$. We prove that $V \cap G^*$ is an $\mathcal{N}$-maximal subgroup of G^*.

Clearly $H \cap C_{G^*}(L(G^*))$ and $K \cap L(G^*)$ are $\mathcal{N}$-injectors of $C_{G^*}(L(G^*))$ and $L(G^*)$ respectively and both of them are contained in $V \cap G^*$. By the former considerations $(H \cap C_{G^*}(L(G^*)))(K \cap L(G^*))$ is an $\mathcal{N}$-maximal subgroup of G^*, hence $V \cap G^*$ is an $\mathcal{N}$-maximal subgroup of G^*.

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